

Equivalent frames in Brans-Dicke theory

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Abstract

We discuss the physical equivalence between the Einstein and Jordan frames in Brans-Dicke theory. The inequivalence of conformal transformed theories is clarified with the help of an old equivalence theorem of Chrisholm's.

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Brans-Dicke theory is a natural generalization of Einstein theory. In Brans-Dicke theory, the effective gravitational constant $G_{eff} = \phi^{-2}$, varies as the Brans-Dicke field ϕ evolves. The evolution of the scalar field slows down the expansion rate of the universe during inflation, and allows nucleation of bubbles to end the inflationary era. There are two conformally related frames, the Einstein frame and the Jordan frame, in Brans-Dicke theory. In the literature, people do not agree with each other about the equivalence of the two frames. Some people think that the two frames are physically equivalent and some people do not think that they are physically equivalent. For example, some people considered the inflationary models in Einstein frame in order to solve equations easily, but they analyzed their final results in Jordan frame because most people insist that the Jordan frame is the physical frame to keep the equivalence principle. In fact, the equivalence principle can be kept in Einstein frame if we use Einstein frame as the physical frame. A dilaton field appears in Kaluza-Klein reduction and string theory also. In string theory, we have the conformally related string frame and Einstein frame. As we all know, the string frame is equivalent to the Einstein frame. How could that the string frame is equivalent to the Einstein frame and the Jordan frame is not equivalent to the Einstein frame? In this letter, we will use the equivalence theorem to address this problem. In general, we can ask if the physics remains the same under an arbitrary change of field variables. Let us first look at the equivalence theorem [1] [2] [3].

Theorem: Let the Lagrangian be known in terms of a set of field variables ϕ , $L[\phi]$. If one expresses the field variables ϕ as nonlinear but local functions of another set of field variables φ (one may think of ϕ as a scalar field for simplicity),

$$\phi = f[\varphi], \quad f[0] = 0, \quad f'[\varphi] \neq 0, \quad (1)$$

one can write down the Lagrangian as $L[\phi] = L[f(\varphi)] = L_t[\varphi]$. The on-mass shell S matrices calculated with $L[\phi]$ and $L_t[\varphi]$ are identical (in making the comparison it may be necessary to introduce appropriate wave-function renormalizations). In other words, both the fields ϕ and φ can be used to describe the same physics.

The important point about the theorem is that we must keep the origin in the field space unchanged when we change the field variables so that we can have the same free field Lagrangian and we do not change the physical quantities, like the mass of the field. In gauge field theories, we give mass to the gauge fields when we shift the Higgs fields by some constants. In the above theorem, when we use the Lagrangian $L_t[\varphi]$, we also need to introduce the Jacobian (the so called Lee-Yang term [4]) due to the change of field variables. The contribution of the Jacobian is necessary to cancel out the most divergent term due to the derivative couplings. The Jacobian is also necessary for an invariant measure in the field space [2] [5]. However, it is not really too important because we set $\delta^4(0) = 0$ when we use the dimensional regularization scheme.

Now let us look at several examples. In string theory, the Einstein frame is related with the string frame by $g_{\mu\nu}^E = e^{-4\phi/(D-2)} g_{\mu\nu}^{str}$ and the dilaton field ϕ is kept invariant. From the equivalence theorem, we know that the string frame is physically equivalent to the Einstein frame.

Suppose we have a free massless scalar field theory,

$$L = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi. \quad (2)$$

Take $\varphi = \phi + \lambda\phi^3$, then the free massless scalar field theory (2) becomes

$$L = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - 3\lambda\phi^2\partial_\mu\phi\partial^\mu\phi - \frac{9}{2}\lambda^2\phi^4\partial_\mu\phi\partial^\mu\phi. \quad (3)$$

Now we get a massless scalar field theory with derivative self-interactions. Since the coupling constant has mass dimension -2 , the theory in terms of ϕ is not power-counting renormalizable. However, this nonrenormalizability is fictitious. We compute the tree diagram for the four point and six point Green's function on shell, the results are zero. The one loop contribution to the $\phi - \phi$ scattering is also zero (the result is proportional to $\delta^4(0)$ which is zero by dimensional regularization). These results are expected because the theory described by (3) is in fact a free theory described by (2).

If we take $\varphi = m_1 e^{\phi/m_1}$ instead, we get

$$\begin{aligned}
L &= -\frac{1}{2}e^{2\phi/m_1}\partial_\mu\phi\partial^\mu\phi \\
&= -\frac{1}{2}\left[1 + \frac{2\phi}{m_1} + \frac{2\phi^2}{m_1^2} + \frac{4\phi^3}{3m_1^3} + \dots\right]\partial_\mu\phi\partial^\mu\phi.
\end{aligned} \tag{4}$$

Again we get a massless scalar field theory with derivative self-interactions. It is not difficult to check that the tree and one-loop diagrams give zero also. Although the exponential transformation does not satisfy the condition of the equivalence theorem, the two fields φ and ϕ describe the same physics for this simple case. This may be understood from the fact that the free field Lagrangians are the same. If we introduce self-interactions, then it is obvious that the exponential transformation introduces mass that changes the free field Lagrangian. Therefore, the exponential transformed theory is not physically equivalent to the original theory in general.

However if we start from a free scalar field of mass m ,

$$L = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2. \tag{5}$$

Take $\varphi = m_1e^{\phi/m_1}$, then Eq. (5) becomes

$$\begin{aligned}
L &= -\frac{1}{2}e^{2\phi/m_1}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2m_1^2e^{2\phi/m_1} \\
&= -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - m^2\phi^2 - \frac{1}{2}m^2m_1^2 - m^2m_1\phi - \frac{2m^2}{3m_1}\phi^3 - \frac{1}{m_1}\phi\partial_\mu\phi\partial^\mu\phi \\
&\quad - \frac{m^2}{3m_1^2}\phi^4 - \frac{1}{m_1^2}\phi^2\partial_\mu\phi\partial^\mu\phi + \dots.
\end{aligned} \tag{6}$$

Now we have a scalar field of mass $\sqrt{2}m$ with self interaction. The physical spectrum is changed. It is easy to verify that the tree diagram for the four point Green's function is not zero on shell.

Now we are ready to discuss the Brans-Dicke theory. The Jordan Brans-Dicke Lagrangian is given by

$$\mathcal{L}_{BD} = -\frac{\sqrt{-\gamma}}{16\pi}\left[\phi\tilde{R} + \omega\gamma^{\mu\nu}\frac{\partial_\mu\phi\partial_\nu\phi}{\phi}\right] - \mathcal{L}_m(\psi, \gamma_{\mu\nu}). \tag{7}$$

If we expand the metric $\gamma_{\mu\nu}$ around the flat metric $\eta_{\mu\nu}$, $\tilde{h}_{\mu\nu} = \gamma_{\mu\nu} - \eta_{\mu\nu}$ does not represent the spin-2 massless graviton. Instead, it is $\rho_{\mu\nu} = \tilde{h}_{\mu\nu} + a\sigma\eta_{\mu\nu}$ that represents the spin-2 massless

graviton. The Lagrangian (7) is conformal invariant under the conformal transformations,

$$g_{\mu\nu} = \Omega^2 \gamma_{\mu\nu}, \quad \Omega = \phi^\alpha, \quad (\alpha \neq \frac{1}{2}), \quad \sigma = \phi^{1-2\alpha}.$$

For the case $\alpha = 1/2$, we make the following transformations

$$g_{\mu\nu} = e^{a\sigma} \gamma_{\mu\nu}, \tag{8a}$$

$$\phi = \frac{8\pi}{\kappa^2} e^{a\sigma}, \tag{8b}$$

where $\kappa^2 = 8\pi G$, $a = \beta\kappa$, $\beta^2 = \frac{2}{2\omega + 3}$. After the conformal transformations (8a) and (8b), we get the Einstein Brans-Dicke Lagrangian

$$\mathcal{L} = \sqrt{-g} \left[-\frac{1}{2\kappa^2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma \right] - \mathcal{L}_m(\psi, e^{-a\sigma} g_{\mu\nu}). \tag{9}$$

Note that the Brans-Dicke Lagrangian is not invariant under the transformations (8a) and (8b). In this frame, $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ represents the spin-2 massless graviton. That's the reason why we think the Einstein frame is the physical frame. The kinetic term of the ϕ field in Eq. (7) can be rewritten as the familiar form $\gamma^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ if we let $\phi = \varphi^2$. As discussed in the examples, the transformations (8a) and (8b) give two physically inequivalent theories if the scalar field ϕ has potential term. For the case discussed here (Eq. (7) and Eq. (9)), the equivalence theorem tells us that the Einstein frame is equivalent to the Jordan frame because the free field Lagrangians are the same. In other words, if we have the same in states $|\gamma_{in}\rangle$ and $|g_{in}\rangle$ and out states $\langle\gamma_{out}|$ and $\langle g_{out}|$, then the cross section calculated in the Jordan frame using the Lagrangian (7) is the same as that calculated in the Einstein frame using the Lagrangian (9). However, the meaning of the equivalence between the Jordan frame and the Einstein frame in the literature is not what we stated here. For example, cosmological models in the Jordan frame are different from those in the Einstein frame. The reason is very simple. If we use the same physical parameters, like the Hubble constant at the present, in both the Jordan frame and the Einstein frame, then the in and out states are not the same in the two frames. In [6], there are some explicit examples to show the difference between the two frames.

The equivalence theorem can tell us if two theories written in terms of different field variables are equivalent. If two theories written in terms of different field variables are not physically equivalent, the theorem can not tell us which theory should be thought as the physical one. If we consider the interactions between gravity and matter fields, then the physical condition, namely the Einstein equivalence principle, rules out the conformal transformations (8a). If we impose a minimal coupling between the gravity and matter in the original theory, then we lose the minimal coupling after the conformal transformations. In other words, we can have covariant conservation law $\nabla^\nu T_{\mu\nu}$ in the original theory, but we do not have the covariant conservation law in the transformed theory.

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